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PARAMETRIC INSTABILITY OF STOCHASTIC COLUMNS

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Abstract—Columns which have stochastically distributed Young's modulus and mass density and are subjected to deterministic periodic axial loadings are considered. The general case of a column supported on a Winkler elastic foundation of random stiffness and also on discrete elastic supports which are also random is considered. Material property fluctuations are modeled as independent one-dimensional univariate homogeneous real random fields in space. In addition to autocorrelation functions or their equivalent power spectral density functions, the input random fields are characterized by scale of fluctuations or variance functions for their second order properties. The foundation stiffness coefficient and the stiffnesses of discrete elastic supports are treated to constitute independent random variables. The system equations of boundary frequencies are obtained using Bolotin's method for deterministic systems. Stochastic FEM is used to obtain the discrete system with random as well as periodic coefficients. Statistical properties of boundary frequencies are derived in terms of input parameter statistics. A complete covariance structure is obtained. The equations developed are illustrated using a numerical example employing a practical correlation structure.

1. INTRODUCTION

There have been a number of studies in the recent past, regarding differential equations with periodic coefficients (Mclachlan, 1947; Yakubovich and Starzhinskii, 1975). A wide variety of real life mechanical systems like rotor-bearing systems, structural systems subjected to vertical ground motion, aircraft structures in a turbulent flow, gun tubes during multiple firing, rocket tanks subjected to longitudinal excitations generated from rocket engines, spinning satellites, etc., are described through such differential equations with periodic coefficients. The time periodic axial loads acting on these systems may induce parametric vibration, a physical phenomenon characterized by unbounded growth of a small perturbation. This parametric vibration causes considerable damage to the mechanical components through critical states like combination resonance, etc., and hence assumes a great research interest. The study of parametric instability of structural systems has, therefore, attracted a considerable amount of research activity over the years. Apart from the excellent monographs by Bolotin (1964) and Evan-Iwanowski (1976), works by Nayfeh and Mook (1979), Dimentberg (1988), Herrmann (1967) and Ibrahim (1985) deserve special mention. The powerful FEM has been used to obtain the boundary frequencies of parametrically excited deterministic systems (Chen and Ku, 1990). However, effects of distributed axial loadings and continuous as well as discrete elastic supports on the dynamic stability characteristics of the systems have not been analysed using FEM. It is worth noting that all these works assume *a priori* that the system properties are deterministic and if at all, only the time varying axial load may be random tempting the usage of terms like "Stochastic Stability". It is only natural to expect that while a perturbation in loads can instigate instability of the system, uncertainties in the system parameters can also similarly affect the system behavior.

It is well known that uncertainty clouds the description of loads, material properties, geometry and boundary conditions in real life structural systems. This uncertainty stems from many factors among which the dominant ones are: (1) Usage of modern construction materials like RCC in civil engineering industry and fiber-reinforced composites in aerospace industry, the material properties of which can be precisely described only in a probabilistic sense; (2) In real mechanical equipment, many factors like non-uniform material density, machining and manufacturing errors, variations in sizes of bolts, rivets, etc., lead to different levels of uncertainty in respect of system parameters; (3) Loadings due to environmental effects are essentially random. As a result, the probabilistic description of strength parameters, failure life, external loadings, etc., has gained much momentum (Shinozuka and Lenoe, 1976; Vanmarcke, 1983; Bolotin, 1989). Of particular interest is the development of theory of local averaging as a result of which newer measures of uncertainty like variance functions, scale of fluctuations, etc., have been evolved (Vanmarcke, 1983). Analysis and design procedures have undergone the necessary modifications, to suit the practice of describing system parameters in a probabilistic sense (Bolotin, 1967; Soong and Cozzarelli, 1976; Schueller and Shinozuka, 1987; Zhu, 1988; Shinozuka, 1987; Augusti *et al.*, 1981; Boyce, 1968; Vom Scheidt and Purkert, 1983; Ibrahim, 1987; Anantha Ramu *et al.*, 1992; Anantha Ramu and Ganesan, 1992b). In this context, the integration of the powerful FEM of structural analysis with the probabilistic mechanics has received a significant amount of impetus recently (Contreras, 1980; Liu *et al.*, 1986; Shinozuka and Deodatis, 1988; Vanmarcke and Grigoriu, 1983; Spanos and Ghanem, 1989; Benaroya and Rehak, 1988; Liaw and Yang, 1991). An efficient version of stochastic FEM wherein the concept of local averaging was coupled with multivariate statistical analysis to yield a powerful computational finite element scheme, has been developed by the present authors (Anantha Ramu and Ganesan, 1991a). This was adopted by the authors (Anantha Ramu and Ganesan, 1991b, 1992a; Anantha Ramu *et al.*, 1991; Sankar *et al.*, 1992) to analyse a variety of stochastically parametered structural systems which are characterized by self-adjoint and nonself-adjoint random eigenvalue problems as well as problems with singularities.

In the present effort, the stochastic FEM is formulated to study the dynamic instability phenomena of parametrically excited stochastic systems. To this end, a column with discrete as well as a continuous Winkler elastic support subjected to deterministic, periodic end thrust and which has stochastically distributed system parameters is considered.

2. SYSTEM DESCRIPTION

A column of span L and second moment of area of cross-section I is considered as shown in Fig. 1. The Young's modulus of elasticity and mass per unit length vary stochastically along its undeformed axis. These variations are identified to be independent one-dimensional univariate homogeneous real stochastic fields in space and are given by

$$E(x) = \bar{E}[1+a(x)], \tag{1}$$

$$m(x) = \bar{m}[1+b(x)], \tag{2}$$

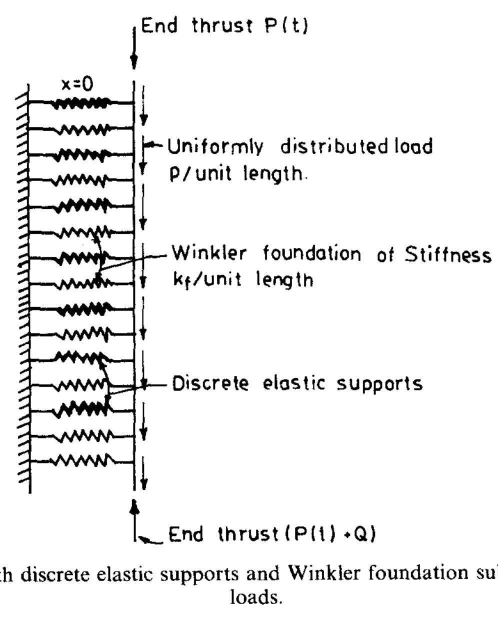

Fig. 1. A column with discrete elastic supports and Winkler foundation subjected to compressive loads.

where x is taken along the undeformed axis of the column, $\bar{E}$ *and* $\bar{m}$ are the respective mean values of Young's modulus and mass density and $a(x)$ and $b(x)$ are two independent zero-mean one-dimensional univariate homogeneous real random fields in space. The random fields are characterized by their respective variances σ_a^2 and σ_b^2, autocorrelation functions $R_{aa}(\tau)$ and $R_{bb}(\tau)$ [or their equivalent power spectral density functions $S_{aa}(f)$ and $S_{bb}(f)$] and scale of fluctuations θ_a and θ_b. In the above, τ is the lag vector and f is the wave frequency of the spectrum of the random fields.

The column subjected to an axial periodic loading given by

$$P(t) = P_0 + P_t \cos \eta t, \tag{3}$$

where η is the axial disturbance frequency, P_0 is the static component and P_t is the time dependent component. P_0, P_t and η are deterministic quantities. Together with this, the column is subjected to an axially distributed deterministic compressive loading of intensity p/unit length.

The support stiffness coefficient of the Winkler foundation, on which the column is resting, is a random variable given by

$$k_f = \bar{k}_f(1+\beta), \tag{4}$$

where k_f is the mean value and β is a zero mean random variable characterized by its variance σ_β^2.

In addition to the Winkler foundation, the column is supported on discrete elastic supports of stiffness k_s at r locations and k_s is a random variable given by

$$k_s = \bar{k}_s(1+s), \tag{5}$$

where $\bar{k}_s$ is the mean value and s is a zero mean random variable characterized by its variance σ_s^2.

3. CONSISTENT FINITE ELEMENT FORMULATION

The governing equations for the stochastic finite element formulation of the problem are now derived directly using the variational principles and Bolotin's method (Bolotin, 1964) for the deterministic case.

The column is divided into NF finite elements with appropriate lengths. The discretization is independent of the stochastic nature of material properties and loadings. The length of a typical element is l_e with end nodes 1 and 2. At each node, translational and rotational d.o.f. are considered and cubic hermitian polynomials are used. Therefore, the transverse displacement at any point is given by

$$W(x,t) = N^e q^e \quad \text{where} \tag{6}$$

$$N^e = [N_1 \quad N_2 \quad N_3 \quad N_4]^e \tag{7}$$

$$q^{e^T} = \{W_1 \quad \theta_1 \quad W_2 \quad \theta_2\}^e, \tag{8}$$

i.e. N^e is the row vector of shape functions and the components of q^e are functions of t, time. Shape functions are in terms of natural coordinates L_1 and L_2.

The total energy stored in the element is given by

$$T^e - U^e = \frac{1}{2}\int_0^{l_e} m(x)\cdot(w_{,t})^2\,\mathrm{d}x - \frac{1}{2}\int_0^{l_e} E(x)\cdot I\cdot(w_{,xx})^2\,\mathrm{d}x \tag{9}$$

$$= \frac{1}{2}q_t^{e^T} M^e q_t^e - \frac{1}{2}q^e\cdot K^e q^e, \tag{10}$$

where

$$M^e = \bar{M}^e + M^e(\Omega)$$

$$= \int_0^{l_e} \bar{m}\cdot N^{e^T}\cdot N^e\,\mathrm{d}x + \bar{m}\int_0^{l_e} b(x)\cdot N^{e^T}N^e\,\mathrm{d}x, \tag{11}$$

$$K^e = \bar{K}^e + K^e(\Omega)$$

$$= \int_0^{l_e} \bar{E}I\cdot N_{xx}^{e^T}N_{xx}^e\,\mathrm{d}x + \int_0^{l_e} \bar{E}Ia(x)\cdot N_{xx}^{e^T}\cdot N_{xx}^e\,\mathrm{d}x. \tag{12}$$

Subscripts t and x denote partial differentiations with respect to time and space and Ω indicates the stochastic nature.

Because of the presence of the distributed axial compressive load the element e is subjected to a uniformly varying axial compression increasing from F_1^e to F_2^e as shown in Fig. 2. The axial compression at any arbitrary section of the element is given by $(F_1^e + px) = (F_1^e + (Q/L)x)$, where x is measured positive in the increasing direction of the axial compression and Q is the total uniformly distributed load. Therefore, the work done by the axial compression in the element, W_c^e is given by

$$W_c^e = \frac{1}{2}\int_0^{l_e} (F_1^e + Qx/L)w_{,x}^2\,\mathrm{d}x$$

$$= \frac{1}{2}F_1^e q^{e^T} K_{GC}^e q^e + \frac{1}{2}Qq^{e^T}K_{GD}^e q^e, \tag{13}$$

where

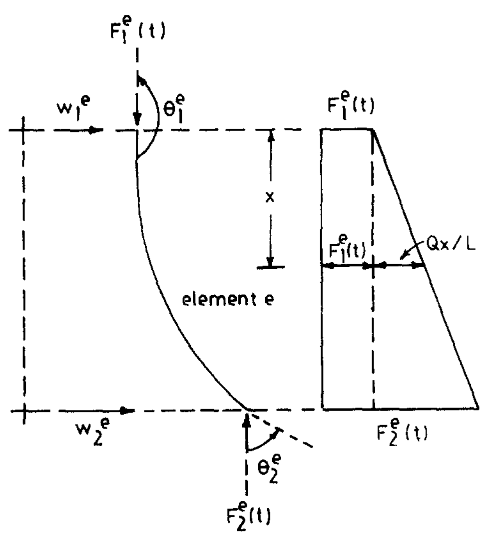

Fig. 2. Distribution of axial compression on the element $-e$.

$$K^e_{GC} = \int_0^{l_e} N_x^{e^T} N_x^e \, dx, \tag{14}$$

$$K^e_{GD} = \int_0^{l_e} N_x^{e^T} (L_2 l_e / L) N_x^e \, dx, \tag{15}$$

F^e_1 is written in the form

$$F^e_1 = P + \alpha^e Q, \tag{16}$$

where α^e is a factor less than 1.0 corresponding to the fraction of the total distributed load acting at the trailing node of the element e.

The work done by the reactive force of the foundation, W^e_f is given by

$$\begin{aligned} W^e_f &= -\frac{1}{2}\int_0^{l_e} k_f w^2 \, dx \\ &= -\frac{1}{2} q^{e^T} K^e_f q_e, \end{aligned} \tag{17}$$

where

$$\begin{aligned} K^e_f &= \bar{K}^e_f + K^e_f(\Omega) \\ &= \int_0^{l_e} N^{e^T} \bar{k}_f N^e \, dx + \int_0^{l_e} N^{e^T} \bar{k}_f \cdot \beta \cdot N^e \, dx. \end{aligned} \tag{18}$$

If the discretization is so made as to have a node at the location of the discrete elastic supports, the work done by such support reactions may be added after all the work quantities due to the distributed forces on each element have been summed up, at the global level, to get the total work done. The work done by the discrete elastic support reaction at a node j, W_{sj} is given by

$$W_{sj} = -\tfrac{1}{2} k_s (w^j)^2 = -\tfrac{1}{2} w^j \bar{k}_s w^j - \tfrac{1}{2} w^j \bar{k}_s \cdot s \cdot w^j, \tag{19}$$

where w^j is the transverse displacement at the jth support. Denoting the total kinetic and elastic strain energies by T and U respectively and further denoting the total work done by W_T, the application of the classical Hamilton's principle to the entire structure yields:

$$\int_{t_1}^{t_2} \delta(T-U) \, dt + \int_{t_1}^{t_2} \delta W_T \, dt = 0, \tag{20}$$

i.e.

$$\begin{aligned} &\int_{t_1}^{t_2} \delta\left[\sum_{e=1}^{NF} \frac{1}{2} q_t^{e^T} \cdot M^e q^e_t\right] dt - \int_{t_1}^{t_2} \delta\left[\sum_{e=1}^{NF} q^{e^T} K^e q^e\right] dt \\ &\quad + \int_{t_1}^{t_2} \delta\left[\sum_{e=1}^{NF} \frac{1}{2} F^e_1 q^{e^T} \cdot K^e_{GC} q^e + \sum_{e=1}^{NF} \frac{1}{2} Q q^{e^T} K^e_{GD} q^e \right. \\ &\quad \left. - \sum_{e=1}^{NF} \frac{1}{2} q^{e^T} K^e_f q^e - \sum_{j=1}^{r} \frac{1}{2} w^j k_s w^j \right] dt = 0. \end{aligned}$$

The summations are made in the sense of finite element assemblage, taking the global displacement vector to be q. Now, if contemporaneous variations of q are taken while integrating the first term by parts, eqn (20) reduces to,

$$\int_{t_1}^{t_2} [-\delta q^{\mathrm{T}} \cdot M q_{tt} - \delta q^{\mathrm{T}} K q + \delta q^{\mathrm{T}} P K_{\mathrm{GC}} q + \delta q^{\mathrm{T}} Q K^*_{\mathrm{GC}} q + \delta q^{\mathrm{T}} Q K_{\mathrm{GD}} q - \delta q^{\mathrm{T}} K_f q - \delta q^{\mathrm{T}} K_s q]\, \mathrm{d}t = 0. \quad (21)$$

It may be noted here that since F_1^e is given by eqn (16), an element modified geometric stiffness matrix is formed as

$$K^{e*}_{\mathrm{GC}} = \alpha^e K^e_{\mathrm{GC}}. \quad (22)$$

As a result, a global modified geometric stiffness matrix denoted by K^*_{GC} is formed by assembling the element matrices K^{e*}_{GC}.

Equation (21) will yield a system of second order differential equations with periodic coefficients of the Mathieu–Hill type. The theory of linear differential equations with periodic coefficients (Bolotin, 1964) suggests that the boundaries between stable and unstable regions can be constructed by periodic solutions of period $2\pi/\eta$ and $4\pi/\eta$. It has also been suggested that the periodic solutions with period $4\pi/\eta$ are of the greatest practical importance. So, we shall seek the periodic solutions with period $4\pi/\eta$ in the following form:

$$q = \tilde{a} \sin \frac{\eta t}{2} + \tilde{b} \cos \frac{\eta t}{2}. \quad (23)$$

Considering the arbitrariness of the variation of q we get,

$$\begin{bmatrix} -K - K_f - K_s + (P_0 - P_t/2) K_{\mathrm{GC}} + Q K_{\mathrm{GD}} + Q K^*_{\mathrm{GC}} & 0 \\ 0 & -K - K_f - K_s + (P_0 + P_t/2) K_{\mathrm{GC}} + Q K_{\mathrm{GD}} + Q K^*_{\mathrm{GC}} \end{bmatrix} \begin{Bmatrix} \tilde{a} \\ \tilde{b} \end{Bmatrix} = \mu \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \tilde{a} \\ \tilde{b} \end{Bmatrix} \quad (24)$$

where $\mu = \eta^2/4$.

This is the equation of boundary frequencies which defines the boundaries between stable and unstable regions. This can also be viewed as a set of linear homogeneous algebraic equations in terms of q which can be obtained first by substituting eqn (23) in eqn (21), then considering the arbitrariness of the variation of q and finally equating the coefficients of sin $(\eta t/2)$ and cos $(\eta t/2)$.

Any total stiffness coefficient is given by

$$\begin{aligned} k_{ij}^{\mathrm{tot}} &= -(\bar{k}_{ij} + k_{ij}(\Omega) + \bar{k}_{ij}^f + k_{ij}^f(\Omega) + \bar{k}_{ij}^s + k_{ij}^s(\Omega)) \\ &= -(\bar{k}_{ij}^{\mathrm{tot}} + k_{ij}^{\mathrm{tot}}(\Omega)) \end{aligned} \quad (25)$$

where $\bar{k}_{ij}^{\mathrm{tot}}$ is the deterministic component, given by

$$\bar{k}_{ij}^{\mathrm{tot}} = -\bar{E}I \int_0^{l_e} N_i''(x) N_j''(x)\, \mathrm{d}x - \int_0^{l_e} N_i(x) N_j(x) \bar{k}_f\, \mathrm{d}x - \bar{k}_{ij}^s \quad (26)$$

and the stochastically fluctuating component $k_{ij}^{\mathrm{tot}}(\Omega)$ is given by

$$k_{ij}^{\text{tot}}(\Omega) = \bar{E}I\int_0^{l_e} a(x)\cdot N_i''(x)N_j''(x)\,\mathrm{d}x - \int_0^{l_e} N_i(x)N_j(x)\bar{k}_f\beta\,\mathrm{d}x - k_{ij}^s(\Omega).$$

In the above and in the sequel primes denote total differentiation in space.

The mean and variance of k_{ij}^{tot} are derived as follows:

Mean value

$$\langle k_{ij}^{\text{tot}}\rangle = -\bar{E}I\int_0^{l_e} N_i''(x)N_j''(x)\,\mathrm{d}x - \int_0^{l_e} N_i(x)\bar{k}_fN_j(x)\,\mathrm{d}x - \bar{k}_{ij}^s$$
$$= \bar{k}_{ij}^{\text{tot}} \tag{27}$$

as $\langle a(x)\rangle = \langle\beta\rangle = \langle s\rangle = 0$.

Variance of k_{ij}^{tot} is derived as,

$$\text{Var}\,(k_{ij}^{\text{tot}}) = \bar{E}^2I^2\int_0^{l_e}\int_0^{l_e}\langle a(\tau_1)\cdot a(\tau_2)\rangle N_i''(\tau_1)\cdot N_j''(\tau_1)$$
$$\cdot N_i''(\tau_2)N_j''(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + \left(\int_0^{l_e}\bar{k}_fN_i(x)N_j\,\mathrm{d}x\right)^2\sigma_\beta^2 + (\bar{k}_{ij}^s)^2\sigma_s^2$$
$$= \bar{E}^2I^2\int_0^{l_e}\int_0^{l_e} R_{aa}(\tau_1-\tau_2)N_i''(\tau_1)\cdot N_j''(\tau_1)N_i''(\tau_2)$$
$$\cdot N_j''(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + \left(\int_0^{l_e}\bar{k}_fN_i(x)N_j(x)\,\mathrm{d}x\right)^2\sigma_\beta^2 + (\bar{k}_{ij}^s)^2\sigma_s^2. \tag{28}$$

Similarly the mass coefficient statistics can be shown to be,

$$\langle m_{ij}\rangle = \bar{m}_{ij} = \bar{m}\int_0^{l_e} N_i(x)N_j(x)\,\mathrm{d}x \tag{29}$$

and

$$\text{Var}\,(m_{ij}) = \bar{m}^2\int_0^{l_e}\int_0^{l_e} R_{bb}(\tau_1-\tau_2)N_i(\tau_1)N_j(\tau_1)N_i(\tau_2)N_j(\tau_2)\cdot\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \tag{30}$$

The covariance between any two stiffness coefficients identified as k_{ij}^{tot} and k_{rs}^{tot} where these are global stiffness coefficients, is derived as follows:

$$\text{Cov}\,(k_{ij}^{\text{tot}}, k_{rs}^{\text{tot}}) = \langle k_{ij}^{\text{tot}}(\Omega)k_{rs}^{\text{tot}}(\Omega)\rangle$$
$$= \left\langle \bar{E}I\int_0^{l_e} a(x)N_i''(x)N_j''(x)\,\mathrm{d}x\cdot\bar{E}I\int_0^{l_2} a(x)N_r''(x)N_s''(x)\right\rangle$$
$$+ (\bar{k}_f)^2\sigma_\beta^2\left(\int_0^{l_1} N_i(x)N_j(x)\,\mathrm{d}x\int_0^{l_2} N_r(x)N_s(x)\,\mathrm{d}x\right) + \sigma_s^2\cdot(\bar{k}_{ij}^s\cdot\bar{k}_{rs}^s), \tag{31}$$

where l_1 and l_2 are the lengths of elements corresponding to the stiffness coefficients. This simplifies to the expression

$$\left\{\bar{E}^2 I^2 \int_0^{l_1}\int_0^{l_2} N_i''(\tau_1)N_j''(\tau_1)N_r''(\tau_2)N_s''(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\right\}\langle E_1(ij)E_2(rs)\rangle$$

$$+(\bar{k}_f)^2\sigma_\beta^2\left(\int_0^{l_1} N_i(x)N_j(x)\,\mathrm{d}x\cdot\int_0^{l_2} N_r(x)N_s(x)\,\mathrm{d}x\right)+\sigma_s^2(\bar{k}_{ij}^s\cdot\bar{k}_{rs}^s). \quad (32)$$

Over each element local averages are formed as follows:

$$E_e=\frac{1}{l_e}\int_0^{l_e} a(x)\,\mathrm{d}x, \quad (33)$$

$$m_e=\frac{1}{l_e}\int_0^{l_e} b(x)\,\mathrm{d}x. \quad (34)$$

These local averages have zero means and their variances are given by:

$$\mathrm{Var}\,(E_e)=\sigma_a^2\Gamma_a(l_e), \quad (35)$$

$$\mathrm{Var}\,(m_e)=\sigma_\beta^2\Gamma_b(l_e), \quad (36)$$

where $\Gamma_a(l_e)$ and $\Gamma_b(l_e)$ are variance functions which are functions of scales of fluctuations θ_a and θ_b. A detailed theory about the scales of fluctuations can be seen in Vanmarcke (1983). As a result, eqn (32) reduces to, after using the variance functions (Anantha Ramu and Ganesan, 1991b):

$$\mathrm{Cov}\,(k_{ij}^{\mathrm{tot}},k_{rs}^{\mathrm{tot}})=\left\{\bar{E}^2 I^2\int_0^{l_1} N_i''(x)N_j''(x)\,\mathrm{d}x\right.$$

$$\left.\times\int_0^{l_2} N_r''(x)N_s''(x)\,\mathrm{d}x\right\}\cdot\frac{\sigma_a^2}{2}[L_0^2\Gamma_a(L_0)-L_1^2\Gamma_a(L_1)$$

$$+L_2^2\Gamma_a(L_2)-L_3^2\Gamma_a(L_3)]+(\bar{k}_f)^2\cdot\sigma_\beta^2\cdot\left(\int_0^{l_1} N_i(x)N_j(x)\,\mathrm{d}x\right.$$

$$\left.\cdot\int_0^{l_2} N_r(x)N_s(x)\,\mathrm{d}x\right)+\sigma_s^2(\bar{k}_{ij}^s\cdot\bar{k}_{rs}^s), \quad (37)$$

where L_0, L_1, L_2 and L_3 are shown in Fig. 3.

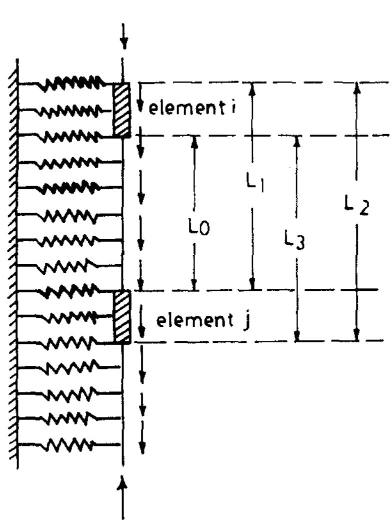

Fig. 3. Correlation parameters corresponding to two arbitrarily located finite elements.

It may be noted that in terms of the correlation functions $r_a(\cdot,\cdot)$ and $r_b(\cdot,\cdot)$, the above equation can be rewritten as:

$$\begin{aligned}\mathrm{Cov}\,(k_{ij}^{\mathrm{tot}}(\Omega),k_{rs}^{\mathrm{tot}}(\Omega)) = \bar{E}^2 I^2 &\int_0^{l_1} N_i''(x)N_j''(x)\,\mathrm{d}x\\ &\cdot\int_0^{l_2} N_r''(x)N_s''(x)\,\mathrm{d}x\\ &\cdot\frac{\sigma_a^2}{2}\left[L_0^2\int_0^{L_0}\int_0^{L_0} r_a(\tau_1-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\right.\\ &-L_1^2\int_0^{L_1}\int_0^{L_1} r_a(\tau_1-\tau_2)\cdot\mathrm{d}\tau_1\,\mathrm{d}\tau_2+L_2^2\int_0^{L_2}\int_0^{L_2} r_a(\tau_1-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\\ &\left.-L_3^2\int_0^{L_3}\int_0^{L_3} r_a(\tau_1-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\right]+(\bar{k}_f)^2\sigma_\beta^2\\ &\cdot\left(\int_0^{l_1} N_i(x)N_j(x)\,\mathrm{d}x\cdot\int_0^{l_2} N_r(x)N_s(x)\,\mathrm{d}x\right)+\sigma_s^2(\bar{k}_{ij}^s\bar{k}_{rs}^s).\end{aligned} \tag{38}$$

Similar expressions can be written for mass coefficients.

4. STATISTICS OF BOUNDARY FREQUENCIES

The equation of boundary frequencies is given by

$$[K^{\mathrm{net}}]\{x\} = \mu[M]\{x\}, \tag{39}$$

where

$$[K^{\mathrm{net}}] = \begin{bmatrix} -K-K_f-K_s+P_0K_{\mathrm{GC}}-P_t/2K_{\mathrm{GC}} & 0 \\ +Q(K_{\mathrm{GD}}+K_{\mathrm{GC}}^*) & -K-K_f-K_s+P_0K_{\mathrm{GC}}+P_t/2K_{\mathrm{GC}} \\ 0 & +Q(K_{\mathrm{GD}}+K_{\mathrm{GC}}^*) \end{bmatrix},$$

$$[M] = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \text{ and } \{x\} = \begin{Bmatrix} \tilde{a} \\ \tilde{b} \end{Bmatrix}.$$

The averaged problem corresponding to eqn (39) is,

$$[\bar{K}^{\mathrm{net}}]\{\bar{x}\} = \bar{\mu}[\bar{M}]\{\bar{x}\}. \tag{40}$$

Since the net stiffness matrix and mass matrix are derived using the stochastic finite element method, consisting of a deterministic component and zero mean fluctuating component, the mean values form the averaged problem.

Using the expressions for covariances between stiffness, mass and geometric stiffness elements, the covariances between the elements of K^{net} are given by their superpositions and it can be noted that the fields $a(x)$ and $b(x)$ are independent.

The covariance matrix can now be constructed as follows:

$$[C_{km}] = \begin{bmatrix} \mathrm{Var}(k_1^{\mathrm{net}}) & \mathrm{Cov}(k_1^{\mathrm{net}}, k_2^{\mathrm{net}})\ldots & \mathrm{Cov}(m_1, k_1^{\mathrm{net}})\ldots & \mathrm{Cov}(m_{n2}, k_2^{\mathrm{net}}) \\ \mathrm{Cov}(k_1^{\mathrm{net}}, k_2^{\mathrm{net}}) & & & \\ \vdots & & & \\ \mathrm{Cov}(m_1, k_1^{\mathrm{net}}) & \mathrm{Cov}(m_1, k_2^{\mathrm{net}})\ldots & \mathrm{Var}(m_1)\ldots & \mathrm{Cov}(m_{n2}, m_1) \\ \vdots & & & \\ \mathrm{Cov}(m_{n2}, k_1^{\mathrm{net}}) & \mathrm{Cov}(m_{n2}, k_2^{\mathrm{net}}) & \mathrm{Cov}(m_{n2}, m_1) & \mathrm{Var}(m_{n2}) \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad (41)$$

where $n2 = 4n^2$; n = global d.o.f. of the structure.

In the above covariance matrix, the submatrices C_{12}, C_{21} become null matrices as the two material property variations are independent, stochastic fields.

As a result, the mean values of boundary frequencies become the boundary frequencies found by solving the unperturbed problem given by eqn (40).

The statistics of boundary frequencies are derived as follows. The perturbations of boundary frequencies can be shown to be

$$\mathrm{d}\mu_i = \sum_{r=1}^{2n}\sum_{s=1}^{2n} \frac{\partial \mu_i}{\partial k_{rs}^{\mathrm{net}}} \mathrm{d}k_{rs}^{\mathrm{net}} + \sum_{r=1}^{2n}\sum_{s=1}^{2n} \frac{\partial \mu_i}{\partial m_{rs}} \mathrm{d}m_{rs}$$

$$= \sum_{r=1}^{2n}\sum_{s=1}^{2n} \left(\frac{\partial \mu_i}{\partial k_{ij}} \mathrm{d}k_{ij} + \frac{\partial \mu_i}{\partial k_{ij}^f} \mathrm{d}k_{ij}^f + \frac{\partial \mu_i}{\partial k_{ij}^s} \mathrm{d}k_{ij}^s \right) + \sum_{r=1}^{2n}\sum_{s=1}^{2n} \frac{\partial \mu_i}{\partial m_{rs}} \mathrm{d}m_{rs}, \quad (42)$$

where

$$\frac{\partial \mu_i}{\partial k_{rs}^{\mathrm{net}}} = x_{ri} x_{si} / x_i^{\mathrm{T}} M x_i, \quad (43)$$

$$\frac{\partial \mu_i}{\partial m_{rs}} = -\mu_i [x_{ri} x_{si} / x_i^{\mathrm{T}} M x_i]. \quad (44)$$

The perturbations of eigenvectors can also be written in a similar manner.

The mean value of any boundary frequency μ_m is derived as follows:

$$\mu_m = \bar{\mu}_m + \sum_{i=1}^{2n}\sum_{j=1}^{2n} \left\{ \frac{\partial \mu_m}{\partial k_{ij}} (k_{ij} - \bar{k}_{ij}) + \frac{\partial \mu_m}{\partial k_{ij}^f} (k_{ij}^f - \bar{k}_{ij}^f) + \frac{\partial \mu_m}{\partial k_{ij}^s} (k_{ij}^s - \bar{k}_{ij}^s) \right\} + \sum_{i=1}^{2n}\sum_{j=1}^{2n} \frac{\partial \mu_m}{\partial m_{ij}} (m_{ij} - \bar{m}_{ij}) \quad (45)$$

$$\mu_m = \bar{\mu}_m + \sum_{i=1}^{2n}\sum_{j=1}^{2n} \left\{ \frac{\partial \mu_m}{\partial k_{ij}} (k_{ij}(\Omega)) + \frac{\partial \mu_m}{\partial k_{ij}^f} (k_{ij}^f(\Omega)) + \frac{\partial \mu_m}{\partial k_{ij}^s} (k_{ij}^s(\Omega)) \right\} + \sum_{i=1}^{2n}\sum_{j=1}^{2n} \frac{\partial \mu_m}{\partial m_{ij}} (m_{ij}(\Omega)), \quad (46)$$

$$\langle \mu_m \rangle = \langle \bar{\mu}_m \rangle + \sum_{i=1}^{2n}\sum_{j=1}^{2n} \left\{ \frac{\partial \mu_m}{\partial k_{ij}^{\mathrm{net}}} \langle k_{ij}^{\mathrm{net}}(\Omega) \rangle + \sum_{i=1}^{2n}\sum_{j=1}^{2n} \frac{\partial \mu_m}{\partial m_{ij}} \langle m_{ij}(\Omega) \rangle. \right.$$

As $\langle k_{ij}^{\mathrm{net}}(\Omega) \rangle = \langle m_{ij}(\Omega) \rangle = 0$,

$$\langle \mu_m \rangle = \bar{\mu}_m. \quad (47)$$

The covariance between any two boundary frequencies is derived as follows:

$$\langle(\mu_m-\bar{\mu}_m)(\mu_n-\bar{\mu}_n)\rangle = \left\langle\left\{\sum_{i=1}^{2n}\sum_{j=1}^{2n}\frac{\partial\mu_m}{\partial k_{ij}^{\text{net}}}(k_{ij}^{\text{net}}(\Omega))+\sum_{i=1}^{2n}\sum_{j=1}^{2n}\frac{\partial\mu_m}{\partial m_{ij}}(m_{ij}(\Omega))\right\}\right.$$
$$\left.\times\left\{\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_n}{\partial k_{rs}^{\text{net}}}(k_{rs}^{\text{net}}(\Omega))+\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_n}{\partial m_{rs}}(m_{rs}(\Omega))\right\}\right\rangle$$
$$=\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_m}{\partial k_{ij}^{\text{tot}}}\frac{\partial\mu_n}{\partial k_{rs}^{\text{tot}}}\langle k_{ij}^{\text{net}}(\Omega),k_{rs}^{\text{net}}(\Omega)\rangle$$
$$+\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_m}{\partial m_{ij}}\frac{\partial\mu_n}{\partial m_{rs}}\langle m_{ij}(\Omega),m_{rs}(\Omega)\rangle \tag{48}$$

as Cov $(k_{ij}^{\text{net}}(\Omega), m_{rs}(\Omega)) = 0$.

Therefore,

$$\text{Var}(\mu_q)=\sigma_{\mu q}^2=\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_q}{\partial k_{ij}^{\text{net}}}\frac{\partial\mu_q}{\partial k_{rs}^{\text{net}}}\text{Cov}(k_{ij}^{\text{net}}(\Omega),k_{rs}^{\text{net}}(\Omega))$$
$$+\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_q}{\partial m_{ij}}\frac{\partial\mu_q}{\partial m_{rs}}\text{Cov}(m_{ij}(\Omega),m_{rs}(\Omega)), \tag{49}$$

$$\text{Cov}(\mu_p,\mu_q)=\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_p}{\partial k_{ij}^{\text{net}}}\frac{\partial\mu_q}{\partial k_{rs}^{\text{net}}}\text{Cov}(k_{ij}^{\text{net}}(\Omega),k_{rs}^{\text{net}}(\Omega))$$
$$+\sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{r=1}^{2n}\sum_{s=1}^{2n}\frac{\partial\mu_p}{\partial m_{ij}}\frac{\partial\mu_q}{\partial m_{rs}}\text{Cov}(m_{ij}(\Omega),m_{rs}(\Omega)). \tag{50}$$

Since

$$k_{ij}^{\text{net}}=k_{ij}^{\text{tot}}+P_0k_{ij}^{\text{GC}}+Qk_{ij}^{\text{GC}*}+Qk_{ij}^{\text{GD}}+A(P_t,k_{ij}^{\text{GC}}), \tag{51}$$

where the function $A(\cdot,\cdot)$ is defined as

$$A(P_t,k_{ij}^{\text{GC}})=-\frac{P_t}{2}k_{ij}^{\text{GC}},\quad i,j\leqslant n$$
$$+\frac{P_t}{2}k_{ij}^{\text{GC}},\quad i,j>n,$$

$$\langle k_{ij}^{\text{net}}\rangle=\bar{k}_{ij}^{\text{net}}=\bar{k}_{ij}^{\text{tot}}+P_0k_{ij}^{\text{GC}}+Qk_{ij}^{\text{GC}*}+Qk_{ij}^{\text{GD}}+A(P_t,k_{ij}^{\text{GC}}). \tag{52}$$

The covariance between $k_{ij}^{\text{net}}(\Omega)$ and $k_{rs}^{\text{net}}(\Omega)$ is given by,

$$C_{ijrs}^{\text{net}}=\langle k_{ij}^{\text{net}}(\Omega),k_{rs}^{\text{net}}(\Omega)\rangle=\langle\{k_{ij}(\Omega)+k_{ij}^f(\Omega)+k_{ij}^s(\Omega)\}\cdot\{k_{rs}(\Omega)+k_{rs}^f(\Omega)+k_{rs}^s(\Omega)\}\rangle$$
$$=\underline{\langle k_{ij}(\Omega),k_{rs}(\Omega)\rangle}+\langle k_{ij}(\Omega)\cdot k_{rs}^f(\Omega)\rangle+\langle k_{ij}(\Omega),k_{rs}^s(\Omega)\rangle+\langle k_{ij}^f(\Omega),k_{rs}(\Omega)\rangle+\underline{\langle k_{ij}^f(\Omega),k_{rs}^f(\Omega)\rangle}$$
$$+\langle k_{ij}^f(\Omega),k_{rs}^s(\Omega)\rangle+\langle k_{ij}^s(\Omega),k_{rs}(\Omega)\rangle+\langle k_{ij}^s(\Omega),k_{rs}^f(\Omega)\rangle+\underline{\langle k_{ij}^s(\Omega),k_{rs}^s(\Omega)\rangle} \tag{53}$$

the underlined terms are nonzero and so,

$$C_{ijrs}^{\text{net}}=\bar{E}^2I^2\int_0^{l_1}\int_0^{l_2}\langle a(\tau_1)a(\tau_2)\rangle N_i''(\tau_1)N_j''(\tau_2)N_r''(\tau_1)N_s''(\tau_2)\cdot\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$
$$+r_{ijrs}^f\sigma_{k_{ij}^f}\sigma_{k_{rs}^f}+r_{ijrs}^s\sigma_{k_{ij}^s}\sigma_{k_{rs}^s}, \tag{54}$$

where r_{ijrs}^f is the correlation coefficient between k_{ij}^f and k_{rs}^f and r_{ijrs}^s is the correlation coefficient between k_{ij}^s and k_{rs}^s. Further

$$-1 \leqslant r^f_{ijrs}, r^s_{ijrs} \leqslant +1.$$

As a result, using eqn (38), C^{net}_{ijrs} is given by

$$C^{\text{net}}_{ijrs} = \left\{ \bar{E}^2 I^2 \int_0^{l_1} N_i'' N_j'' \,\mathrm{d}x \int_0^{l_2} N_r'' N_s'' \,\mathrm{d}x \right\} \cdot \frac{\sigma_a^2}{2} \left\{ L_0^2 \int_0^{L_0} \int_0^{L_0} r_a(x_1 - x_2) \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right.$$
$$- L_1^2 \int_0^{L_1} \int_0^{L_1} r_a(x_1 - x_2) \,\mathrm{d}x_1 \,\mathrm{d}x_2 + l_2^2 \int_0^{L_2} \int_0^{L_2} r_a(x_1 - x_2) \,\mathrm{d}x_1 \,\mathrm{d}x_2 - L_3^2$$
$$\left. \cdot \int_0^{L_3} \int_0^{L_3} r_a(x_1 - x_2) \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right\} + r^f_{ijrs} \sigma_{k^f_{ij}} \sigma_{k^f_{rs}} + r^s_{ijrs} \sigma_{k^s_{ij}} \sigma_{k^s_{rs}}. \quad (55)$$

The expressions used to evaluate the covariance of eigenvector elements are very similar to those above since each eigenvector element is also perturbed about its averaged value as eigenvalues are perturbed. The covariance matrix of eigensolution can thus be constructed using the above formulae.

5. SOLUTION OF THE DETERMINISTIC CASE

The expressions developed in the foregoing constitute the solution of the general case of a stochastic column resting on continuous and discrete elastic supports. The deterministic axial loading is pulsating at the end and is uniformly distributed along the length of the column. Such a general treatment becomes possible because the finite element method is employed. Any comparison with the results of standard analytical procedures, however, would be possible only for a simplified deterministic case.

It is easy to verify that in the absence of all random quantities, all discrete and continuous elastic supports and the distributed axial load equations (39) reduce to the simplified case of the finite element analysis of a deterministic column under pulsating loads examined by Chen and Ku (1990). The last cited work has demonstrated the accuracy of the finite element method of analysis of the deterministic case by comparing the results with those of standard analytical procedures. This exercise is therefore not repeated in this paper and in the following numerical study only the stochastic case will be examined.

6. NUMERICAL EXAMPLE

A column with simply-supported end conditions is considered as shown in Fig. 4. The Young's modulus as well as the stiffness of the Winkler foundation are treated as random. The mean value of E is given by

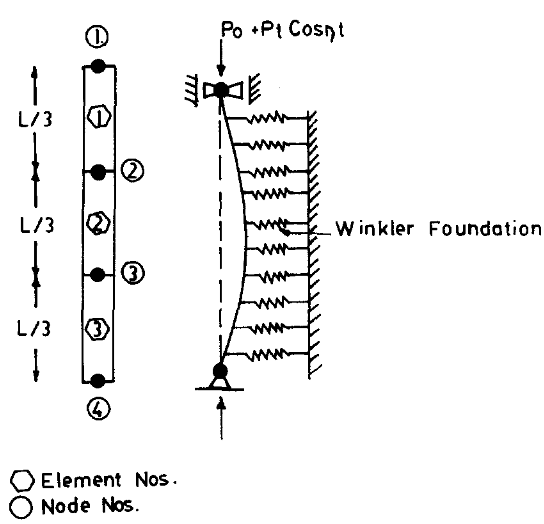

Fig. 4. Example problem.

Table 1. Variances of boundary frequencies when E is random

Input variance $\sigma_{E_i}^2 \times 10^{+2}$	Var (μ_1) $\times 10^{+4}$	Var (μ_2)	Var (μ_3) $\times 10^{+2}$	Var (μ_4)	Var (μ_5) $\times 10^{+3}$	Var (μ_6) $\times 10^{-4}$	Var (μ_7) $\times 10^{+2}$
1.0	3.7708	0.4628	1.5588	1392.2894	3.1580	8.9718	2.2488
2.0	7.5417	0.9257	3.1177	2784.5789	6.3161	17.9436	4.4976
3.0	11.3125	1.3885	4.6765	4176.8683	9.4741	26.9153	6.7464
4.0	15.5083	1.8513	6.2354	5569.1577	12.6321	35.8871	8.9952
5.0	18.8542	2.3141	7.7942	6961.4472	15.7901	44.8589	11.2440
6.0	22.6250	2.7770	9.3531	8353.7366	18.9482	53.8307	13.4927
7.0	26.3958	3.2398	10.9120	9746.0261	22.1062	62.8025	15.7415
8.0	30.1667	3.7026	12.4708	11 138.3155	25.2642	71.7742	17.9903
9.0	33.9376	4.1655	14.0296	12 530.6049	28.4222	80.7460	20.2391
10.0	37.7084	4.6283	15.5885	13 922.8944	31.5803	89.7178	22.4879

$$\bar{E} = 2.1 \times 10^5 \,\mathrm{N\,mm^{-2}}.$$

The mass density is

$$7.83 \times 10^{-9} \,\mathrm{N\,sec\,mm^{-4}},$$

and the length of the column is 7.35 m. Further, $P = 1$ N.

The stochastic process $a(x)$ representing the fluctuating components of modulus of elasticity can have any correlation structure but for illustrative purposes is represented by the exponential type correlation function.

The correlation function is given by

$$r(\tau) = \mathrm{e}^{-|\tau|/b}; \quad b = \text{constant}.$$

The corresponding variance function is given by (Vanmarcke, 1983):

$$\Gamma(U) = 2\left(\frac{b}{u}\right)^2 \left(\frac{U}{b} - 1 + \mathrm{e}^{-U/b}\right).$$

First, k_f is set equal to zero and uncorrelated E values are considered.

The variances of boundary frequencies which characterize the stochasticity of boundary frequencies are obtained for different input variances and are given in Table 1. Covariances between boundary frequencies which describe their inter-statistical-dependence are given in Tables 2 and 3. Now, k_f is considered to be 17900 N m. Again, variances of boundary frequencies for different input variances of both the Young's modulus and Winkler elastic support are plotted in Fig. 5 and covariances between boundary frequencies are plotted in Fig. 6. In all these numerical results, input variance σ_E^2, which is a function of scale of

Table 2. Covariances between boundary frequencies when E is random

Input variance $\sigma_{E_i}^2 \times 10^{+2}$	Cov (μ_1, μ_3) $\times 10^{+15}$	Cov (μ_1, μ_5) $\times 10^{+7}$	Cov (μ_{12}, μ_9) $\times 10^{+18}$
1.0	9.7272	9.9823	1.5989
2.0	19.4543	19.9646	3.1978
3.0	29.1814	29.9488	4.6341
4.0	38.9086	39.9291	6.3957
5.0	48.6358	49.9113	7.7236
6.0	58.3629	59.8936	9.2683
7.0	68.0901	69.8759	11.6504
8.0	77.8172	79.8582	12.7914
9.0	87.5444	89.8404	13.9024
10.0	97.2715	99.8227	15.4471

Table 3. Covariances between boundary frequencies when E is random

Input variance $\sigma^2_{E_i} \times 10^{+2}$	Cov (μ_1, μ_7) $\times 10^{+17}$	Cov (μ_2, μ_{12}) $\times 10^{+14}$	Cov (μ_1, μ_9) $\times 10^{+7}$
1.0	2.2148	0.9848	−3.1357
2.0	4.4296	1.9096	−6.2714
3.0	6.6495	2.9545	−9.4071
4.0	8.8593	3.9393	−12.5428
5.0	11.0795	4.9241	−15.6785
6.0	13.2991	5.9089	−18.8142
7.0	15.5244	6.8938	−21.9499
8.0	17.7186	7.8786	−25.0856
9.0	19.9364	8.8634	−28.2213
10.0	22.1591	9.8482	−31.3570

fluctuation θ_a is varied to result in a parametric study with reference to the scale of fluctuation.

It may be noted that the correlation model employed in this example can be interpreted as exactly representing the "First-order autoregressive models" and the "Markov chain models". These two models are identified in the published literature as the practical engineering stochastic models giving excellent performance in modeling the field data. Further it is the constant b that determines the memory length and so the parametric study that follows presents the effect of its variation through σ_{Ei} on the system response moments. Further, this model contains one memory in excess of ideal white noise models of infinite power. One solution bound is always given by white noise field only. But, the total power is infinite in the case of a white noise field, which is impractical. However, the practical finite power stochastic fields can always be derived from ideal white noise fields with the least effort by allowing only one memory, which is done here. Thus, a practical field model is considered.

The influence of different types of input correlation models on the variability of eigenvalues of a nonself-adjoint stochastic system is examined by the authors in an earlier work (1992).

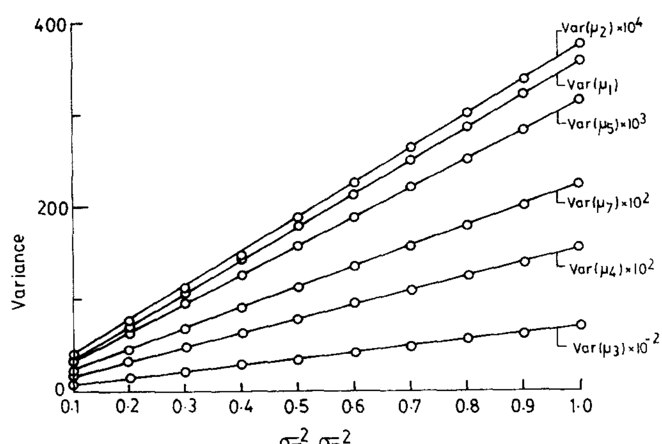

Fig. 5. Variances of boundary frequencies.

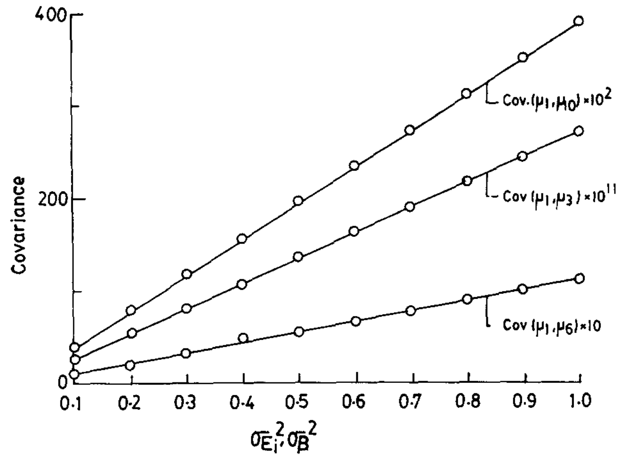

Fig. 6. Variances between boundary frequencies.

7. CONCLUSIONS

The stochastic finite element method is formulated to solve the random system of boundary frequencies of an elastically supported stochastic column subjected to deterministic periodic axial loadings. The formulation makes the finite element discretization independent of the stochastic nature of the material property fluctuations and stiffnesses of elastic supports. The stochastic fields are defined in terms of means, variances and scale of fluctuations. The stochastic finite element method so developed enables the derivation of complete statistics of boundary frequencies and corresponding modes directly in terms of input parameter statistics. The computational efficiency of the method is demonstrated through a numerical example. Extension of the foregoing procedure to problems in two and three dimensions appears to be straightforward.

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